

Energy Optimal Control for Quantum System Evolving on $SU(1,1)$

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This paper discusses the energy optimal control problem for the class of quantum systems that possess dynamical symmetry of $SU(1,1)$, which are widely studied in various physical problems in the quantum theory. Based on the maximum principle on Lie group, the complete set of optimal controls are analytically obtained, including both normal and abnormal extremals. The results indicate that the normal extremal controls can be expressed by the Weierstrass elliptic function, while the abnormal extremal controls can only be constant functions of time t .

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I. INTRODUCTION

During the past two decades, optimization techniques have been extensively applied to design external control fields to manipulate the evolution of quantum mechanical systems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. A typical application is to force the states to approach the priori prescribed targets as closely as possible, including both bounded [1, 2, 3, 4] and unbounded [5] situations. Also, optimization theory can be applied to improve the efficiency of desired quantum state transitions, e.g., the evolution time [6, 7, 8, 9, 10, 11] and the energy consumed by the controls [12, 13] are most interesting. For some low dimensional quantum systems that evolve on compact Lie groups, one can find analytical solutions for such optimization problems with bounded [7, 12] or unbounded [6] controls. However, in the higher dimensional situations, numerical algorithms have to be applied.

In this paper, we explore the optimal steering problem for the class of quantum systems whose evolution operators can be described by the $SU(1,1)$ matrices. The underlying system is modelled by an evolution equation of the form [14]:

$$\frac{d}{dt}X(t) = [A + u(t)B]X(t), \quad X(0) = I_2, \quad (1)$$

where $X(t)$ is a two dimensional special pseudo-unitary matrix; $u(t)$ is real function of time t , which is the control input of the system; A and B are arbitrary matrices that can be expressed as the linear combination of K_x , K_y and K_z , which are the generators of the Lie algebra $su(1,1)$ and can be identified as follows:

$$K_x = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad K_y = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad K_z = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \quad (2)$$

In [15], Jurdjevic has taken the initial steps in the problems of optimal control for the special case when $A = K_x$ and $B = K_z$ (the corresponding properties of the optimal controls also can be found for the systems evolving on the homomorphic groups $SO(2,1)$ and $SL(2, \mathbb{R})$ in [15] and [16]). Unlike the case of $SU(2)$ [17], however, as will be seen in Section II, system (1) usually can't be transformed into the special form such that $A = K_x$ and $B = K_z$. Thus, it is natural for us to consider the general case in detail.

With specific realizations and representations of the Lie algebra $su(1,1)$ introduced, system (1) can be used to describe various of quantum dynamical processes, e.g., the superfluid system under Bose realization [18], the harmonic oscillator under xp -realization [19], the $SU(1,1)$ coherent states under irreducible unitary representation with respect to positive discrete series [20].

The optimal control problem to be considered in this paper is formulated as follows. Given an arbitrary target evolution matrix X_f in $SU(1,1)$, we wish to find a control function $u(t)$ that can steer the evolution matrix associated

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with system (1) from its initial state I_2 to some desired final state X_f , and meanwhile, minimize the quadratic cost function

$$J(u) = \int_0^T u(t)^2 dt, \quad (3)$$

where T is the final time. The quadratic cost index given in (3) measures the energy consumed during the steering process between the initial I_2 and the terminal X_f . Since the remarkable difference between the quantum systems and the classical systems is that the evolutions of the former may be disturbed by the decoherence phenomenon. This is practical because that the increasing of the intensity of the electromagnetic fields, which are used to control the evolution of the coupled quantum system, tend to induce relaxation and decoherence phenomena.

Based on the maximum principle for systems evolving on Lie group [21], explicit forms of the control functions with respect to both normal and abnormal extremals will be derived analytically. The problem considered here can be viewed as the noncompact prolongation of the $SU(2)$ case presented in [12]. However, according to quantum theory and group representation theory [22], the noncompact $SU(1, 1)$ Lie group has only infinite dimensional unitary representations, and hence the associated evolution operator (or propagator) corresponding to $X(t)$ is always infinite dimensional, which describes the transition between two quantum states defined in an infinite dimensional Hilbert space. The dynamics of the $SU(1, 1)$ coherent states is a typical example [20]. This makes the derivation quite different from and far more complicated than that in the case of $SU(2)$.

The balance of this paper is organized as follows. In Section II, some useful results including controllability properties and the maximum principle with respect to the quantum system evolving on Lie group $SU(1, 1)$ are introduced. In Section III, we discuss the optimal steering problem with respect to the abnormal extremals. Properties of the abnormal optimal control function are characterized. In Section IV, the control functions corresponding to the normal extremals are derived analytically for all possible cases. In Section V, two examples are provided for illustration. Finally, conclusions are drawn in Section VI.

II. PRELIMINARIES ON THE QUANTUM CONTROL SYSTEM ON THE LIE GROUP $SU(1, 1)$

With the three generators K_x , K_y and K_z of the Lie algebra $su(1, 1)$ in (2), any given evolution matrix X associated with system (1) can be written as

$$X = e^{\alpha K_z} e^{\beta K_y} e^{\gamma K_x}, \quad (4)$$

where $-2\pi < \alpha, \gamma \leq 2\pi$, $0 \leq \beta < \infty$. The commutation relations between K_x , K_y and K_z are

$$[K_x, K_y] = -K_z, \quad [K_y, K_z] = K_x, \quad [K_z, K_x] = K_y. \quad (5)$$

With the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle M, N \rangle = 2\text{Tr}(MN^\dagger), \quad (6)$$

where N^\dagger is the Hermitian conjugation of N , it can be verified that K_x , K_y and K_z form an orthonormal basis of the Lie algebra $su(1, 1)$. Accordingly, the drift term A and the control term B in system (1) can be expressed by linear combinations of the three generators K_x , K_y and K_z as follows:

$$A = \langle A, K_x \rangle K_x + \langle A, K_y \rangle K_y + \langle A, K_z \rangle K_z, \quad (7)$$

$$B = \langle B, K_x \rangle K_x + \langle B, K_y \rangle K_y + \langle B, K_z \rangle K_z. \quad (8)$$

In this paper, we say M is pseudo-orthogonal to N when $\langle M, N^\dagger \rangle = 0$. And M will be called elliptic (hyperbolic, parabolic) if $\langle M, M^\dagger \rangle$ is negative (positive, zero). Accordingly, the matrices in the Lie algebra $su(1, 1)$ are separated into three different types. Since for any matrix $P \in SU(1, 1)$, we have

$$\langle PMP^{-1}, (PMP^{-1})^\dagger \rangle = 2\text{Tr}(PMP^{-1}PMP^{-1}) = 2\text{Tr}(MM) = \langle M, M^\dagger \rangle, \quad (9)$$

i.e., none of the changes of coordinates by the $SU(1, 1)$ transformations will alter the type of an $su(1, 1)$ matrix. Thus, as mentioned in Section I, system (1) usually can't be transformed into the special case such that $A = K_x$ (hyperbolic) and $B = K_z$ (elliptic).

We assume system (1) is controllable in this paper, this ensures that the optimal control problem stated in the introduction section is always solvable. System (1) is said to be controllable on $SU(1, 1)$ if for any given target

evolution matrix $X_f \in SU(1, 1)$ there always exists at least one control $u(t)$ such that $X(u; T) = X_f$ for some time T . The problems of the controllability for systems evolving on Lie groups have received a great deal of attention in the past decades (see e.g., [23, 24]). As for the systems evolving on the noncompact Lie group $SU(1, 1)$, a sufficient and necessary condition is provided in [25] and can be summarized as follows.

Theorem II.1 ([25])

1. If A and B are linearly dependent, then system (1) is uncontrollable on $SU(1, 1)$;
2. If A and B are linearly independent, then system (1) is controllable if and only if the set $\{u \in \mathbb{R} \mid \langle A + uB, A^\dagger + uB^\dagger \rangle < 0\}$ is nonempty.

To obtain the optimal control functions, we will make use of the well-known maximum principle on Lie groups [21, 24], which, under the assumption that system (1) is controllable, can be summarized for the $SU(1, 1)$ case as follows [21].

Theorem II.2 Assume that system (1) is controllable, if $u^o(t)$ is an optimal control that minimizes the quadratic index given in (3) and $X^o(t)$ is the corresponding optimal trajectory of (1). Then, there exists a constant matrix $S \in su(1, 1)$ and a nonnegative real number λ_o , not both zero, such that for almost everywhere (a.e.) the Hamiltonian function

$$H(S; \lambda_o; u; X_o(t)) = \langle S, X^o(t)^{-1} [A + u(t)B] X^o(t) \rangle + \frac{1}{2} \lambda_o u(t)^2 \quad (10)$$

is minimized with respect to u by $u^o(t)$.

The above theory immediately provides a necessary condition for the optimality. The problems with respect to $\lambda_o \neq 0$ are called normal, otherwise are called abnormal.

III. OPTIMAL CONTROL FUNCTION FOR THE ABNORMAL EXTREMAL

In this section, we will investigate the abnormal case. For this purpose, the following basic properties on $su(1, 1)$ are useful.

Lemma III.1 For arbitrary pair of matrices M and N in $su(1, 1)$, the following relations hold:

$$[[M, N], M] = \langle M, N^\dagger \rangle M - \langle M, M^\dagger \rangle N, \quad (11)$$

$$[[M, N], N] = \langle N, N^\dagger \rangle M - \langle M, N^\dagger \rangle N. \quad (12)$$

proof Since (11) is equivalent to (12) because of symmetry, we only need to proof (11). If M and N are linearly dependent, then there exists a real constant $k \neq 0$ such that $M = kN$. Therefore,

$$[[M, N], M] = [[kN, N], kN] = 0, \quad (13)$$

and

$$\langle M, N^\dagger \rangle M - \langle M, M^\dagger \rangle N = \langle kN, N^\dagger \rangle kN - \langle kN, kN^\dagger \rangle N = 0, \quad (14)$$

i.e., the relation in (11) holds. Thus, we only need to consider the case when M and N are linearly independent, i.e., $[M, N] \neq 0$. Since

$$\langle [M, N]^\dagger, M \rangle = \langle [M, N], M^\dagger \rangle = \langle MN, M^\dagger \rangle - \langle NM, M^\dagger \rangle = 0, \quad (15)$$

and

$$\langle [M, N]^\dagger, N \rangle = \langle [M, N], N^\dagger \rangle = \langle MN, N^\dagger \rangle - \langle NM, N^\dagger \rangle = 0, \quad (16)$$

M , N and $[M, N]^\dagger$ are still linearly independent. Thus we can express $[[M, N], M]$ as

$$[[M, N], M] = \mu_1 M + \mu_2 N + \mu_3 [M, N]^\dagger, \quad (17)$$

for some constants μ_1 , μ_2 and μ_3 . Taking the inner product with $[M, N]^\dagger$ in (17), we obtain:

$$\mu_3 \langle [M, N]^\dagger, [M, N]^\dagger \rangle = 0. \quad (18)$$

Eq. (18) implies that $\mu_3 = 0$, thus we can rewrite (17) as:

$$[[M, N], M] = \mu_1 M + \mu_2 N. \quad (19)$$

Taking the inner product of (19) with M^\dagger , we get

$$\mu_1 \langle M, M^\dagger \rangle + \mu_2 \langle N, M^\dagger \rangle = 0. \quad (20)$$

Since

$$\langle [[M, N], M], N^\dagger \rangle = 2\text{Tr}([M, N], M)N = 2\text{Tr}([M, N][M, N]) = \langle [M, N], [M, N]^\dagger \rangle, \quad (21)$$

take the inner product of (19) with N^\dagger , we have

$$\langle [M, N], [M, N]^\dagger \rangle = \mu_1 \langle M, N^\dagger \rangle + \mu_2 \langle N, N^\dagger \rangle. \quad (22)$$

Make use of the equality $\langle [M, N], [M, N]^\dagger \rangle = \langle M, N^\dagger \rangle^2 - \langle M, M^\dagger \rangle \langle N, N^\dagger \rangle$ (see the proof in [25]), from (20) and (22) we have

$$\mu_1 = \langle M, N^\dagger \rangle, \quad \mu_2 = -\langle M, M^\dagger \rangle.$$

Lemma III.2 *If M and N are linearly independent and the set $\{u \in \mathbb{R} \mid \langle M + uN, M^\dagger + uN^\dagger \rangle < 0\}$ is nonempty, then M , N and $[M, N]$ form a basis in $su(1, 1)$.*

proof Assume that $[M, N]$ can be linearly expressed by M and N , i.e. there exist two real number λ_1 and λ_2 satisfy

$$[M, N] = \lambda_1 M + \lambda_2 N. \quad (23)$$

On the one hand, making communications with M and N respectively, one can obtain

$$[[M, N], M] = -\lambda_2 [M, N] = -\lambda_1 \lambda_2 M - \lambda_2^2 N, \quad (24)$$

$$[[M, N], N] = \lambda_1 [M, N] = \lambda_1^2 M + \lambda_1 \lambda_2 N. \quad (25)$$

On the other hand, notice that (11) and (12) would imply

$$[[M, N], M] = \langle M, N^\dagger \rangle M - \langle M, M^\dagger \rangle N, \quad (26)$$

$$[[M, N], N] = \langle N, N^\dagger \rangle M - \langle M, N^\dagger \rangle N. \quad (27)$$

Compare the coefficients of (24) and (25) and those of (26) and (27) respectively, we obtain

$$\lambda_1^2 = \langle N, N^\dagger \rangle, \quad \lambda_1 \lambda_2 = -\langle M, N^\dagger \rangle, \quad \lambda_2^2 = \langle M, M^\dagger \rangle. \quad (28)$$

A straightforward computation, using (28), shows that, for every $u \in \mathbb{R}$,

$$\begin{aligned} & \langle M + uN, M^\dagger + uN^\dagger \rangle \\ &= \langle M, M^\dagger \rangle + 2u \langle M, N^\dagger \rangle + u^2 \langle N, N^\dagger \rangle \\ &= \lambda_2^2 - 2\lambda_1 \lambda_2 u + \lambda_1^2 u^2 \\ &= (\lambda_2 - \lambda_1 u)^2 \geq 0. \end{aligned} \quad (29)$$

This contradicts with the condition that the set $\{u \in \mathbb{R} \mid \langle M + uN, M^\dagger + uN^\dagger \rangle < 0\}$ is nonempty. So M , N and $[M, N]$ are linearly independent.

With the properties on the Lie algebra $su(1, 1)$ obtained above, we can now draw the conclusion for the abnormal case as follows.

Theorem III.3 Assume that the system (1) is controllable. If $\langle B, B^\dagger \rangle \neq 0$, then the control function $u(t) = -\langle A, B^\dagger \rangle / \langle B, B^\dagger \rangle$, a.e., is the only abnormal extremal for the optimization problem under consideration. Otherwise, there is no abnormal extremal.

proof If $\lambda_o = 0$, the Hamiltonian function may be rewritten as

$$H(S; \lambda_o; u; X_o(t)) = \langle S, X^o(t)^{-1}[A + u(t)B]X^o(t) \rangle. \quad (30)$$

Since the minimization condition indicates that $H(S; \lambda_o; u; X_o(t))$ is a.e. minimized with respect to $u(t)$ by $u_o(t)$. To obtain the optimal control function $u_o(t)$, we differentiate $H(S; \lambda_o; u; X_o(t))$ with respect to u and set the result equal to zero, which yields

$$\langle S, X^o(t)^{-1}BX^o(t) \rangle = 0, \quad \text{a.e.} \quad (31)$$

By continuity, (31) implies that

$$\langle S, X^o(t)^{-1}BX^o(t) \rangle \equiv 0. \quad (32)$$

Differentiating (32) with respect to t , using (1), we can obtain

$$\langle S, X^o(t)^{-1}[A, B]X^o(t) \rangle = 0. \quad (33)$$

Differentiating (33) with respect to t again, we have

$$\langle S, X^o(t)^{-1}[[A, B], A]X^o(t) \rangle + u(t) \langle S, X^o(t)^{-1}[[A, B], B]X^o(t) \rangle = 0. \quad (34)$$

Utilizing (11), (12) and (31), we can recast (34) to

$$(\langle A, B^\dagger \rangle + u(t) \langle B, B^\dagger \rangle) \langle S, X^o(t)^{-1}AX^o(t) \rangle = 0, \quad \text{a.e.} \quad (35)$$

Notice that, according to Lemma III.2, A , B and $[A, B]$ are linearly independent. If the equality $\langle S, X^o(t)^{-1}AX^o(t) \rangle = 0$ holds, then combining (31) and (33) we can draw conclusion that S must be zero, which is a contradiction. Therefore Eq.(35) implies that

$$\langle A, B^\dagger \rangle + u(t) \langle B, B^\dagger \rangle = 0, \quad \text{a.e.} \quad (36)$$

If $\langle B, B^\dagger \rangle = 0$, from Eq.(36) we have $\langle A, B^\dagger \rangle = 0$, which contradicts the assumption that system (1) is controllable (see [25]). Thus, there is no abnormal extremal when $\langle B, B^\dagger \rangle = 0$. If $\langle B, B^\dagger \rangle \neq 0$, then from (36) we have $u(t) = -\langle A, B^\dagger \rangle / \langle B, B^\dagger \rangle$, a.e.

Remark: The target evolution matrices that can be achieved directly by the abnormal control $u = -\langle A, B^\dagger \rangle / \langle B, B^\dagger \rangle$ are in the one dimensional Lie group corresponding to the Lie subalgebra of $su(1,1)$ generated by $A - \langle A, B^\dagger \rangle / \langle B, B^\dagger \rangle B$, which can never fill up the whole Lie group of $SU(1,1)$. Consequently, abnormal extremal exists only when the target evolution matrix X_f is of the form $\exp[c(A - \langle A, B^\dagger \rangle / \langle B, B^\dagger \rangle B)]$ for some real constant c . In order to solve the optimal steering problem subject to the terminal condition $X(T) = X_f$, where X_f is an arbitrary matrix taken from $SU(1,1)$, the candidates can only be normal extremals.

IV. OPTIMAL CONTROL FUNCTION FOR THE NORMAL EXTREMAL

In this section we explore the normal extremal control functions. According to Theorem II.1, here we assume that A and B are linearly independent and meanwhile the set $\{u \in \mathbb{R} | \langle A + uB, A^\dagger + uB^\dagger \rangle < 0\}$ is nonempty throughout this section, to guarantee the controllability of the system.

One can obtain the normal extremal, according to Theorem II.2, by minimizing the Hamiltonian function $H(S; \lambda_o; u; X_o(t))$ as a quadratic function of u . After normalizing $\lambda_o = 1$, by continuity, the necessary condition for candidate optimal controls can be expressed as

$$u(t) = -\langle S, X_o^{-1}(t)BX_o(t) \rangle, \quad \text{a.e.} \quad (37)$$

where the matrix S is an element in the Lie algebra $su(1,1)$. We introduce the following two auxiliary variables in the succeeding discussion:

$$\begin{aligned} u_A(t) &= -\langle S, X_o(t)^{-1}AX_o(t) \rangle, \\ u_C(t) &= -\langle S, X_o(t)^{-1}[A, B]X_o(t) \rangle. \end{aligned} \quad (38)$$

In order to follow standard notations, we rewrite the normal extremal $u(t)$ in (37) as $u_B(t)$. Differentiate u_A , u_B and u_C with respect to the time t , respectively, and make use of (1), (11) and (12), we can obtain (a.e.)

$$\dot{u}_A = u_B u_C, \quad (39)$$

$$\dot{u}_B = -u_C, \quad (40)$$

$$\dot{u}_C = \alpha u_A - \beta u_B + \gamma u_A u_B - \alpha u_B^2, \quad (41)$$

where

$$\alpha = \langle A, B^\dagger \rangle, \quad \beta = \langle A, A^\dagger \rangle, \quad \gamma = \langle B, B^\dagger \rangle. \quad (42)$$

From (39)-(41), it is easy to verify the following conclusion for the normal extremal u_B and the two auxiliary variables u_A and u_C .

Theorem IV.1 *The following two quantities are conserved along the normal extremal trajectories, i.e.,*

$$u_A + \frac{1}{2}u_B^2 = c_1, \quad (43)$$

$$\frac{1}{2}\gamma u_A^2 - \alpha u_A u_B - \frac{1}{2}u_C^2 - \beta u_A = c_2, \quad (44)$$

for some constants c_1 and c_2 .

According to Theorem IV.1, the initial and the final values of u_A , u_B and u_C should satisfy (a.e.)

$$\begin{aligned} u_A(0) + \frac{1}{2}u_B(0)^2 &= u_A(T) + \frac{1}{2}u_B(T)^2, \\ \frac{1}{2}\gamma u_A(0)^2 - \alpha u_A(0)u_B(0) - \frac{1}{2}u_C(0)^2 - \beta u_A(0) &= \frac{1}{2}\gamma u_A(T)^2 - \alpha u_A(T)u_B(T) - \frac{1}{2}u_C(T)^2 - \beta u_A(T), \end{aligned} \quad (45)$$

where

$$u_A(0) = -\langle S, A \rangle, \quad u_B(0) = -\langle S, B \rangle, \quad u_C(0) = -\langle S, [A, B] \rangle, \quad (46)$$

and

$$\begin{aligned} u_A(T) &= -\langle S, X(T)^{-1}AX(T) \rangle, \\ u_B(T) &= -\langle S, X(T)^{-1}BX(T) \rangle, \\ u_C(T) &= -\langle S, X(T)^{-1}[A, B]X(T) \rangle, \end{aligned} \quad (47)$$

The matrix S in Eqs.(46) and (47) then can be viewed as parameter matrix, which has to be chosen to match the final condition $X(T) = X_f$.

Table I. The controllability of system (1) with respect to the values of α , β and γ .

The values of α , β , and γ			the controllability of system (1)
$\alpha = 0$	$\gamma < 0$	$\beta = 0$	uncontrollable
		$\beta \neq 0$	controllable
	$\gamma = 0$		uncontrollable
	$\gamma > 0$	$\beta < 0$	controllable
		$\beta \geq 0$	uncontrollable
$\alpha \neq 0$	$\gamma \leq 0$		controllable
	$\gamma > 0$	$\beta \leq 0$	controllable
		$\beta > 0$	uncontrollable
		$\alpha^2 - \beta\gamma \leq 0$	
		$\alpha^2 - \beta\gamma > 0$	controllable

Making use of (43) and (44), from (39)-(41), we can obtain the following differential equation for the candidate optimal control $u(t)$ given in (37)

$$(\dot{u})^2 = \frac{\gamma}{4}u^4 + \alpha u^3 + (\beta - \gamma c_1)u^2 - 2\alpha c_1 u + \gamma c_1^2 - 2\beta c_1 - 2c_2, \text{ a.e.} \quad (48)$$

where $u(0) = u_B(0)$. We will show, in the following, that the candidate optimal control function can be analytically solved from (48) in terms of the Weierstrass function.

Since the involved system is assumed to be controllable to ensure that the optimal steering problem has solutions, one only need to consider the controllable situations accordingly. Table I shows the controllability properties of system (1) in different cases. (see [25] for details). There are three different cases, which need to be taken into account, depending on the values of α and γ .

1. *Case $\alpha = 0$ and $\gamma \neq 0$.*

In this case, the drift term A of system (1) is pseudo-orthogonal to the control term B while the latter is not parabolic. Accordingly, Eq.(48) can be simplified as (a.e.)

$$(\dot{u})^2 = \frac{\gamma}{4}u^4 + (\beta - \gamma c_1)u^2 + \gamma c_1^2 - 2\beta c_1 - 2c_2. \quad (49)$$

In order to obtain the explicit form of the optimal control function from (49), by a variable replacement $x = \frac{\gamma}{4}u^2 + \frac{\beta - \gamma c_1}{3}$, we rewrite this differential equation as (a.e.)

$$(\dot{x})^2 = 4x^3 - g_2x - g_3, \quad (50)$$

where $g_2 = \frac{1}{3}[(\beta - \gamma c_1)^2 + 3(\beta^2 + 2\gamma c_2)]$ and $g_3 = \frac{1}{27}[(\beta - \gamma c_1)^2 - 9(\beta^2 + 2\gamma c_2)]$.

From the classical theory of elliptic functions (see, e.g., [26, 27]), it is well known that the above differential equation is satisfied by the Weierstrass function $\mathcal{G}(\cdot; g_2, g_3)$ when the discriminant $g_2^3 - 27g_3^2$ is nonzero. Therefore, one can express the candidate optimal control function as (a.e.)

$$u(t) = \pm \sqrt{\frac{4}{\gamma} \left(\mathcal{G}(t + a; g_2, g_3) - \frac{\beta - \gamma c_1}{3} \right)}, \quad (51)$$

when $g_2^3 - 27g_3^2 \neq 0$, where $a = \mathcal{G}^{-1} \left(\frac{\gamma}{4}u_B(0)^2 + \frac{\beta - \gamma c_1}{3}; g_2, g_3 \right)$. The sign of the above candidate optimal control function $u(t)$ turns at the point when $u(t)$ cross the t axis.

Consider the exceptional situations that the discriminant of (50) is zero, i.e.,

$$g_2^3 - 27g_3^2 = (\beta^2 + 2\gamma c_2)[(\beta^2 + 2\gamma c_2) - (\beta - \gamma c_1)^2]^2 = 0, \quad (52)$$

It is easy to see that either (i) $\beta^2 + 2\gamma c_2 = 0$ or (ii) $(\beta^2 + 2\gamma c_2) - (\beta - \gamma c_1)^2 = 0$.

For the case (i), a further use of (44) leads to that $\beta^2 + 2\gamma c_2 = (\gamma u_A - \beta)^2 - \gamma u_C^2 = 0$. Thus, if $\gamma < 0$, we have $u_A = \frac{\beta}{\gamma}$, $u_B = u_B(0)$ and $u_C = 0$, which determines the candidate optimal control function by $u(t) = u_B(0)$ (a.e.). If $\gamma > 0$, from (49), we have (a.e.)

$$(\dot{u})^2 = \frac{\gamma}{4}[u^2 + \frac{2}{\gamma}(\beta - \gamma c_1)]^2, \quad (53)$$

whose solutions can be expressed as follows.

- If $\beta - \gamma c_1 < 0$, then (a.e.)

$$u(t) = \begin{cases} \pm \sqrt{\frac{2(\gamma c_1 - \beta)}{\gamma}} \tanh \left(\sqrt{\frac{\gamma c_1 - \beta}{2}} t + a \right), & \text{when } |u_B(0)| < \sqrt{\frac{2(\gamma c_1 - \beta)}{\gamma}}; \\ \pm \sqrt{\frac{2(\gamma c_1 - \beta)}{\gamma}} \coth \left(\sqrt{\frac{\gamma c_1 - \beta}{2}} t + a \right), & \text{when } |u_B(0)| > \sqrt{\frac{2(\gamma c_1 - \beta)}{\gamma}}, \end{cases} \quad (54)$$

where

$$a = \begin{cases} \text{Ar th} \left(\pm \sqrt{\frac{\gamma}{2(\gamma c_1 - \beta)}} u_B(0) \right), & \text{when } |u_B(0)| < \sqrt{\frac{2(\gamma c_1 - \beta)}{\gamma}}; \\ \text{Ar cth} \left(\pm \sqrt{\frac{\gamma}{2(\gamma c_1 - \beta)}} u_B(0) \right), & \text{when } |u_B(0)| > \sqrt{\frac{2(\gamma c_1 - \beta)}{\gamma}}. \end{cases} \quad (55)$$

- If $\beta - \gamma c_1 = 0$, then (a.e.)

$$u(t) = \pm \frac{2}{\sqrt{\gamma(t + a)}}, \quad (56)$$

where $a = \pm \frac{\sqrt{\gamma}}{2} u_B(0)$.

- If $\beta - \gamma c_1 > 0$, then (a.e.)

$$u(t) = \pm \sqrt{\frac{2(\gamma c_1 - \beta)}{\gamma}} \tan \left(\sqrt{\frac{\gamma c_1 - \beta}{2}} t + a \right), \quad (57)$$

where $a = \arctan \left(\pm \sqrt{\frac{\gamma}{2(\gamma c_1 - \beta)}} u_B(0) \right)$.

For the case (ii), the equation (49) can be simplified as (a.e.)

$$(\dot{u})^2 = \frac{\gamma}{4} [u^2 + \frac{4}{\gamma} (\beta - \gamma c_1)] u^2. \quad (58)$$

If $\gamma < 0$, clearly, the above differential equation has no nontrivial solution other than $u(t) \equiv 0$ when $\beta - \gamma c_1 < 0$. When $\beta - \gamma c_1 > 0$, one can immediately obtain the optimal control function from (58) as (a.e.)

$$u(t) = \frac{2\sqrt{\frac{\gamma c_1 - \beta}{\gamma}}}{\text{ch} \left(\pm \sqrt{\beta - \gamma c_1} t + a \right)}, \quad (59)$$

where $a = \text{Ar ch} \left(\frac{2\sqrt{\frac{\beta - \gamma c_1}{\gamma}}}{u_B(0)} \right)$. If $\gamma > 0$, then from (58), one can obtain the candidate optimal control function as follows.

- If $\beta - \gamma c_1 < 0$, then (a.e.)

$$u(t) = 2\sqrt{\frac{\gamma c_1 - \beta}{\gamma}} \sec \left(\pm \sqrt{\gamma c_1 - \beta} t + a \right), \quad (60)$$

where $a = \text{arcsec} \left(\frac{1}{2} \sqrt{\frac{\gamma}{\gamma c_1 - \beta}} u_B(0) \right)$.

- If $\beta - \gamma c_1 > 0$, then (a.e.)

$$u(t) = \frac{2\sqrt{\frac{\beta - \gamma c_1}{\gamma}}}{\text{sh} \left(\pm \sqrt{\beta - \gamma c_1} t + a \right)}, \quad (61)$$

where $a = \text{Ar sh} \left(\frac{2\sqrt{\frac{\beta - \gamma c_1}{\gamma}}}{u_B(0)} \right)$.

Remark: In the case of $\alpha = 0$ and $\gamma \neq 0$, the optimal control function can be expressed by the Weierstrass elliptic function only when the discriminant $g_2^3 - 27g_3^2$ of (50) is nonzero. When $g_2^3 - 27g_3^2 = 0$, the optimal control function is reduced to elementary functions. In comparison with the case that $g_2^3 - 27g_3^2 \neq 0$, the case of $g_2^3 - 27g_3^2 = 0$ occurs with only a probability of zero. The similar arguments also can be made in the following two cases.

2. Case $\alpha \neq 0$ and $\gamma = 0$.

In this case, control term B of system (1) is parabolic. Accordingly, Eq.(48) can be simplified as (a.e.)

$$(\dot{u})^2 = \alpha u^3 + \beta u^2 - 2\alpha c_1 u - 2\beta c_1 - 2c_2, \quad (62)$$

With $x = \frac{1}{4}\alpha u + \frac{1}{12}\beta$, (62) can be recast as (a.e.)

$$(\dot{x})^2 = 4x^3 - g_2x - g_3, \quad (63)$$

where $g_2 = \frac{1}{12}(\beta^2 + 6\alpha^2 c_1)$ and $g_3 = \frac{1}{216}(18\alpha^2 \beta c_1 + 27\alpha^2 c_2 - \beta^3)$.

If the discriminant of (63) $g_2^3 - 27g_3^2 \neq 0$, one can again obtain the candidate optimal control function in terms of Weierstrass elliptic function (a.e.)

$$u(t) = \frac{1}{\alpha} \left[4\mathcal{G}(t + a; g_2, g_3) - \frac{\beta}{3} \right], \quad (64)$$

where $a = \mathcal{G}^{-1} \left(\frac{1}{4}[\alpha u_B(0) + \frac{\beta}{3}]; g_2, g_3 \right)$.

Otherwise, $4x^3 - g_2x - g_3$ has repeated zeros. Accordingly, the candidate optimal control function can be obtained from (62) or (63) as follows.

- If $g_3 < 0$, then

$$u(t) = -\frac{6}{\alpha} \sqrt[3]{g_3} \left(\frac{1 + \exp(\pm \sqrt{-6\sqrt[3]{g_3}} t + a)}{1 - \exp(\pm \sqrt{-6\sqrt[3]{g_3}} t + a)} \right)^2 + \frac{4}{\alpha} \sqrt[3]{g_3} - \frac{\beta}{3\alpha}, \quad (65)$$

$$\text{where } a = \ln \frac{\sqrt{\frac{1}{4}(\alpha u_B(0) + \frac{\beta}{3}) - \sqrt[3]{g_3}} - \sqrt{-\frac{3}{2}\sqrt[3]{g_3}}}{\sqrt{\frac{1}{4}(\alpha u_B(0) + \frac{\beta}{3}) - \sqrt[3]{g_3}} + \sqrt{-\frac{3}{2}\sqrt[3]{g_3}}}.$$

- If $g_3 = 0$, then

$$u(t) = \frac{4}{\alpha(t \pm a)^2} - \frac{\beta}{3\alpha}, \quad (66)$$

$$\text{where } a = \pm \frac{2}{\sqrt{\alpha u_B(0) + \frac{\beta}{3}}}.$$

- If $g_3 > 0$, then

$$u(t) = \frac{6}{\alpha} \sqrt[3]{g_3} \tan \left(\pm \sqrt{\frac{3}{2}\sqrt[3]{g_3}} t + a \right) + \frac{4}{\alpha} \sqrt[3]{g_3} - \frac{\beta}{3\alpha}, \quad (67)$$

$$\text{where } a = \arctan \sqrt{\frac{\alpha u_B(0) + \frac{\beta}{3} - 4\sqrt[3]{g_3}}{6\sqrt[3]{g_3}}}.$$

3. Case $\alpha \neq 0$ and $\gamma \neq 0$.

In this case, none of the drift term A and the control term B is parabolic. Let

$$\begin{cases} f(x) = \frac{\gamma}{4}x^4 + \alpha x^3 + (\beta - \gamma c_1)x^2 - 2\alpha c_1 x + \gamma c_1^2 - 2\beta c_1 - 2c_2, \\ g_2 = \frac{\gamma}{4}c + \frac{\alpha^2}{2}c_1 + \frac{1}{12}(\beta - \gamma c_1)^2, \\ g_3 = \frac{\gamma(\beta - \gamma c_1)}{24}c - \frac{\alpha^2(\beta - \gamma c_1)c_1}{24} - \frac{(\beta - \gamma c_1)^3}{216} - \frac{\alpha^2 \gamma c_1^2}{16} - \frac{\alpha^2}{16}c, \end{cases} \quad (68)$$

where $c = \gamma c_1^2 - 2\beta c_1 - 2c_2$. Let x_1, x_2, x_3 and x_4 denote the roots of the equation $f(x) = 0$.

When $g_2^3 - 27g_3^2 \neq 0$, it can be verified that $x_i \neq x_j$ ($1 \leq i < j \leq 4$). It is known that the solution of Eq.(49) still can be written down explicitly in terms of the Weierstrass function (see [26] chapter XX), which is given by (a.e.)

$$u(t) = x_0 + \frac{6f'(x_0)}{24\mathcal{G}(t + a; g_2, g_3) - f''(x_0)}, \quad (69)$$

where $a = \mathcal{G}^{-1} \left(\frac{f'(x_0)}{4(u_B(0) - x_0)} + \frac{1}{24}f''(x_0); g_2, g_3 \right)$ and $x_0 \in \{x_1, x_2, x_3, x_4\}$.

When $g_2^3 - 27g_3^2 = 0$, the polynomial $f(x)$ in (68) has repeated zeros. Then, there are four different situations accordingly.

1) $f(x)$ has only one 2-fold zero.

In this case, without loss of generality, we can assume that $x_4 \neq x_1 = x_2 \neq x_3 \neq x_4$. Accordingly, (49) can be rewritten as (a.e.)

$$(\dot{u})^2 = \frac{\gamma}{4}(u - x_1)^2[u^2 - (x_3 + x_4)u + x_3x_4]. \quad (70)$$

From (70), one can compute the corresponding candidate optimal control function as (a.e.)

$$u(t) = \frac{2\gamma(x_1 - x_3)(x_1 - x_4)}{|\gamma(x_3 - x_4)| \sin \left(\pm \frac{1}{2} \sqrt{-\gamma(x_1 - x_3)(x_1 - x_4)} t + a \right) - \gamma(2x_1 - x_3 - x_4)} + x_1, \quad (71)$$

where $a = \arcsin \frac{\gamma(2x_1 - x_3 - x_4)(u_B(0) - x_1) + 2\gamma(x_1 - x_3)(x_1 - x_4)}{(u_B(0) - x_1)|\gamma(x_3 - x_4)|}$, when $\gamma(x_1 - x_3)(x_1 - x_4) < 0$; and

$$u(t) = \frac{\gamma(2x_1 - x_3 - x_4) + 2 \left(\exp(\pm \frac{1}{2} \sqrt{\gamma(x_1 - x_3)(x_1 - x_4)} t + a) - \frac{\gamma(2x_1 - x_3 - x_4)}{2\sqrt{\gamma(x_1 - x_3)(x_1 - x_4)}} \right) \sqrt{\gamma(x_1 - x_3)(x_1 - x_4)}}{\left(\exp(\pm \frac{1}{2} \sqrt{\gamma(x_1 - x_3)(x_1 - x_4)} t + a) - \frac{\gamma(2x_1 - x_3 - x_4)}{2\sqrt{\gamma(x_1 - x_3)(x_1 - x_4)}} \right)^2 - \gamma} + x_1, \quad (72)$$

where $a = \ln \left(\frac{\sqrt{\gamma(u_B(0)-x_1)^2 + \gamma(2x_1-x_3-x_4)(u_B(0)-x_1) + \gamma(x_1-x_3)(x_1-x_4)} + \sqrt{\gamma(x_1-x_3)(x_1-x_4)}}{u_B(0)-x_1} \right. \\ \left. + \frac{\gamma(2x_1-x_3-x_4)}{2\sqrt{\gamma(x_1-x_3)(x_1-x_4)}} \right),$

when $\gamma(x_1 - x_3)(x_1 - x_4) > 0$.

2) $f(x)$ has two different 2-fold zeros.

In this case, without loss of generality, it can be assumed that $x_1 = x_2 \neq x_3 = x_4$. Accordingly, (49) can be rewritten as (a.e.)

$$(\dot{u})^2 = \frac{\gamma}{4}(u - x_1)^2(u - x_3)^2. \quad (73)$$

If $\gamma > 0$ (there is no nontrivial solution for $\gamma < 0$), from (73), we have

$$u(t) = -\frac{(x_1 - x_3) \exp(\pm \frac{\gamma}{2}(x_1 - x_3)t + a)}{\exp(\pm \frac{\gamma}{2}(x_1 - x_3)t + a) - 1}, \quad (74)$$

where $a = \ln \frac{u_B(0)-x_1}{u_B(0)-x_3}$.

3) $f(x)$ has one 3-fold zero.

It can be assumed, accordingly, that $x_1 = x_2 = x_3 \neq x_4$. Then, (49) can be rewritten as (a.e.)

$$(\dot{u})^2 = \frac{\gamma}{4}(u - x_1)^3(u - x_4). \quad (75)$$

The corresponding candidate optimal control function is given by (a.e.)

$$u(t) = \frac{4\gamma(x_1 - x_4)}{\gamma^2(x_1 - x_4)^2(a \pm \frac{1}{2}t)^2 - 4\gamma} + x_1, \quad (76)$$

where $a = -\frac{2}{\gamma(x_1-x_4)(u_b(0)-x_1)} \sqrt{\gamma(u_b(0)-x_1)^2 + \gamma(x_1-x_4)(u_b(0)-x_1)}$.

4) $f(x)$ has one 4-fold zero.

In this case, we can assume that $x_1 = x_2 = x_3 = x_4$. Then, (49) can be rewritten as (a.e.)

$$(\dot{u})^2 = \frac{\gamma}{4}(u - x_1)^4. \quad (77)$$

If $\gamma > 0$ (there is no real solution for $\gamma < 0$), from (77) one can obtain the candidate optimal control function as (a.e.)

$$u(t) = \frac{1}{a \pm \frac{\sqrt{\gamma}}{2}t} + x_1, \quad (78)$$

where $a = \frac{1}{u_B(0)-x_1}$.

V. EXAMPLES

In this section, we give two examples for illustration.

Example 1: Consider the case when $A = K_x + 2K_z$, $B = K_x$ and the target evolution matrix has the form $X_f = e^{\theta K_z}$ ($\theta \in \mathbb{R}$).

It can be checked that the abnormal optimal control $u_{\text{abnormal}}(t) = -\frac{\langle A, B^\dagger \rangle}{\langle B, B^\dagger \rangle} = -1$ steers system (1) from the initial state I_2 to the final state X_f in time $T_f = (\theta \bmod 4\pi)/2$, with the performance measure given by

$$J(u_{\text{abnormal}}) = \int_0^{(\theta \bmod 4\pi)/2} u_{\text{abnormal}}(t) dt = (\theta \bmod 4\pi)/2. \quad (79)$$

Actually, taking the matrix S as

$$S = s_z K_z, \quad (80)$$

we have

$$\begin{aligned}
H(S; \lambda_o; u; X_o(t)) &= \langle S, X^o(t)^{-1} [A + u(t)B] X^o(t) \rangle \\
&= \langle s_z K_z, e^{-2tK_z} [K_x + 2K_z + uK_x] e^{2tK_z} \rangle \\
&= \langle s_z K_z, (1+u)[\cos(2t)K_x - \sin(2t)K_y] + 2K_z \rangle \\
&\equiv 2s_z,
\end{aligned} \tag{81}$$

which is minimized with respect to u by the abnormal optimal control $u_{\text{abnormal}}(t)$.

Example 2: Suppose that the system (1) is given by $A = K_z$ and $B = -K_x + K_y$. Consider the optimal steering problem with the terminal condition $X_f = e^{-2K_x + 2K_y}$.

Write the matrix S as

$$S = s_x K_x + s_y K_y + s_z K_z. \tag{82}$$

In order to achieve target evolution matrix optimally, according to the Eqs. (46) and (47), the coefficients s_x , s_y and s_z in (82) are required to satisfy

$$s_z = \frac{\sinh(2\sqrt{2})}{\sqrt{2}[\cosh(2\sqrt{2}) - 1]}(s_x + s_y). \tag{83}$$

Fig 1 shows the resulting evolutions.

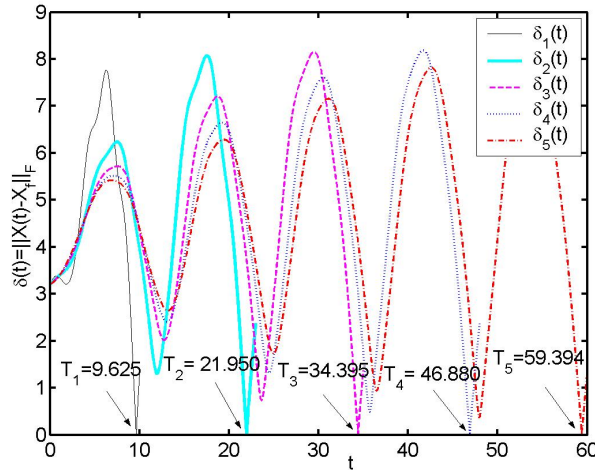


FIG. 1: The Distance Between the State $X(t)$ and the Target X_f at time t . A numerical treatment shows that when the two independent parameters (s_x, s_y) take the values of $(-2.09895, 0.99801)$, $(-0.844738, 0.406815)$, $(-0.523766, 0.254595)$, $(-0.379568, 0.185591)$, $(-0.297326, 0.145922)$, \dots , the target evolution matrix $X_f = e^{-2K_x + 2K_y}$ can be reached at time $T_1 = 9.625$, $T_2 = 21.950$, $T_3 = 34.395$, $T_4 = 46.880$, $T_5 = 59.394$, \dots . Where the norm $\|X\|_F$ is defined as $\|X\|_F = \sqrt{\sum_{i,j} |X_{ij}|^2}$.

Correspondingly, the control functions given by

$$\begin{aligned}
|u_1(t)| &= \sqrt{2\mathcal{G}(t + 0.5740 - 1.2502i; 9.8362, 4.4871) + 2.4990} \\
&= \frac{1.0547}{\sqrt{\mathcal{G}(t + 0.5740; 9.8362, 4.4871) + 1.2495}},
\end{aligned} \tag{84}$$

$$\begin{aligned}
|u_2(t)| &= \sqrt{2\mathcal{G}(t + 0.6705 - 2.0412i; 3.6531, 1.3108) + 1.2418} \\
&= \frac{0.3487}{\sqrt{\mathcal{G}(t + 0.6705; 3.6531, 1.3108) + 0.6209}},
\end{aligned} \tag{85}$$

$$\begin{aligned}
|u_3(t)| &= \sqrt{2\mathcal{G}(t + 0.7051 - 2.5221i; 2.6390, 0.8203) + 0.9956} \\
&= \frac{0.2045}{\sqrt{\mathcal{G}(t + 0.7051; 2.6390, 0.8203) + 0.4978}},
\end{aligned} \tag{86}$$

$$|u_4(t)| = \sqrt{2\mathcal{G}(t + 0.7247 - 2.8643i; 2.2391, 0.6435) + 0.8954} \\ = \frac{0.1442}{\sqrt{\mathcal{G}(t+0.7247; 2.2391, 0.6435) + 0.4477}}, \quad (87)$$

$$|u_5(t)| = \sqrt{2\mathcal{G}(t + 0.7370 - 3.1118i; 2.0254, 0.5543) + 0.8416} \\ = \frac{0.1115}{\sqrt{\mathcal{G}(t+0.7370; 2.0254, 0.5543) + 0.4208}}, \quad (88)$$

⋮

as shown in Fig 2, are candidate optima.

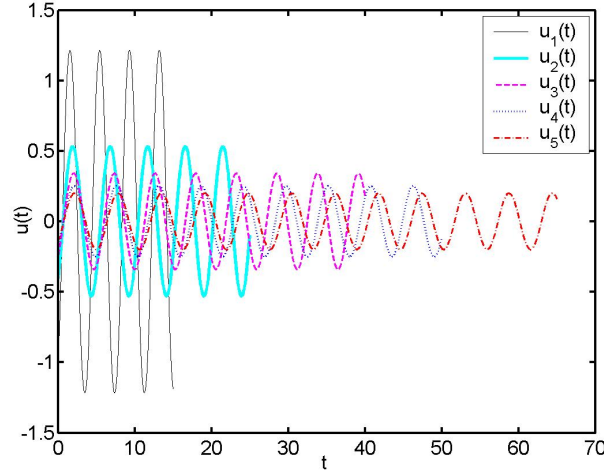


FIG. 2: The Candidate Optimal Control Functions.

The corresponding performance measures are shown in Fig 3. It is easy to observe that there is a tradeoff between the consumed time T and the cost index $J(u)$.

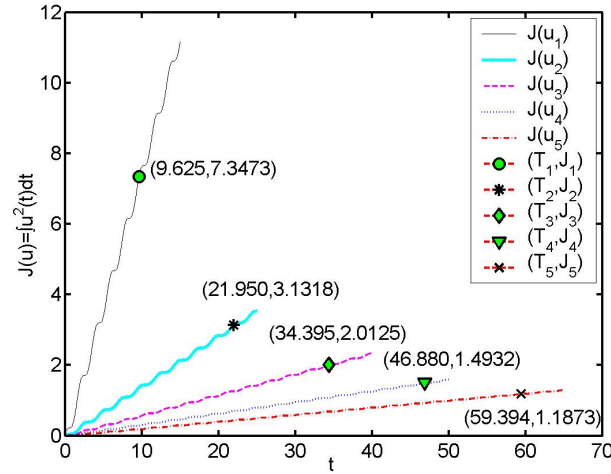


FIG. 3: The Performance Measures Corresponding to the Candidate Optimal Controls. Where $J(u_1) = 7.3473$, $J(u_2) = 3.1318$, $J(u_3) = 2.0125$, $J(u_4) = 1.4932$, $J(u_5) = 1.1873$, \dots .

According to the decomposition algorithm given in [14], piecewise constant control laws can be designed to achieve the target evolution $X_f = e^{-2K_x + 2K_y}$ as well. One possible control law is given by

$$\bar{u}(t) = \begin{cases} 0, & \text{when } t \in [0, t_1 + 2n_1\pi]; \\ \frac{c}{\sqrt{2}}, & \text{when } t \in [t_1 + 2n_1\pi, t_1 + t_2 + 2n_1\pi]; \\ 0, & \text{when } t \in (t_1 + t_2 + 2n_1\pi, 2t_1 + t_2 + 2(n_1 + n_2)\pi], \end{cases} \quad (89)$$

where $n_1, n_2 \in \mathbb{N}^+$, $c > 1$ and

$$\begin{cases} t_1 = 2 \operatorname{arccot}(-c \operatorname{cth} \sqrt{2} - \sqrt{c^2 \operatorname{cth}^2 \sqrt{2} - 1}), \\ t_2 = \frac{2}{\sqrt{c^2 - 1}} \operatorname{Ar} \operatorname{cth} \left(\frac{1}{\sqrt{c^2 - 1}} \sqrt{c^2 \operatorname{cth}^2 \sqrt{2} - 1} \right). \end{cases}$$

The corresponding performance measure is given by

$$J(\bar{u}) = \frac{c^2}{\sqrt{c^2 - 1}} \operatorname{Ar} \operatorname{cth} \left(\frac{1}{\sqrt{c^2 - 1}} \sqrt{c^2 \operatorname{cth}^2 \sqrt{2} - 1} \right). \quad (90)$$

It can be verified that $J(\bar{u})$ increases monotonously with the increase of c . Since

$$\lim_{c \rightarrow 1} J(\bar{u}) = \frac{1}{\sqrt{\operatorname{cth}^2 \sqrt{2} - 1}} \approx 1.9351, \quad (91)$$

as a comparison, the performance measures $J(u_4)$ and $J(u_5)$ are approximately 13 and 39 percent, respectively, less than $J(\bar{u})$.

VI. CONCLUSION

In order to minimize the decoherence effect, energy optimal control problem for the quantum systems evolving on the noncompact Lie group $SU(1, 1)$ is taken into account in this paper. We showed that explicit expressions for the optimal control functions can be obtained analytically. To minimize the considered quadratic performance measure, the control functions with respect to abnormal extremals are constant functions of time t , while those with respect to normal extremals are expressed by the Weierstrass elliptic function.

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